

## An Initial Value Method for Integral Operators: I-Complex Eigenvalues

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A method is described for the calculation of the eigenvalues of general integral operators. Several classical results from functions of a complex variable and the theory of integral equations are combined with a recent technique for converting Fredholm integral equations into an initial-valued system of differential equations. The technique applies to both symmetric and nonsymmetric kernels.

### 1. INTRODUCTION

In this paper we shall combine some of the classical results of Fredholm [1] and complex variable theory [2] with the recent techniques [3, 4] of initial-value or Cauchy systems for integral equations with general kernels. The combining of these modern and classical techniques leads to a flexible and accurate method for the calculation of complex eigenvalues. The method is of particular interest in that it applies to both symmetric and nonsymmetric kernels. Related work for real eigenvalues only is contained in [5].

### 2. CAUCHY SYSTEM

We wish to solve the homogeneous equation

$$U(t) = \lambda \int_0^1 k(t, y) U(y) dy. \quad (1)$$

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Fredholm [1] showed that the resolvent  $K(t, y; \lambda)$  of the kernel  $k(t, y)$  could be expressed as

$$K(t, y; \lambda) = \frac{D(t, y; \lambda)}{D(\lambda)}, \tag{2}$$

where  $D(t, y; \lambda)$  and  $D(\lambda)$  are *entire* functions of  $\lambda$ . He also showed that  $D(t, y; \lambda)$  and  $D(\lambda)$  are related by the differential equation

$$D_\lambda(\lambda) = - \int_0^1 D(y', y'; \lambda) dy' \tag{3}$$

and that

$$D(0) = 1. \tag{4}$$

The subscript in (3) denotes differentiation with respect to  $\lambda$ . The eigenvalues are the roots of the equation

$$D(\lambda) = 0. \tag{5}$$

At a simple root  $\lambda = \lambda^*$  an eigenfunction is given by  $D(t, y; \lambda^*)$  for any  $y$ .

In recent work [3, 4] it has been shown that the resolvent kernel  $K(t, y; \lambda)$  satisfies the Cauchy system

$$K_\lambda(t, y; \lambda) = \int_0^1 K(t, y'; \lambda) K(y', y; \lambda) dy', \quad 0 \leq t, y \leq 1, \tag{6}$$

subject to the initial condition

$$K(t, y; 0) = k(t, y). \tag{7}$$

If desired this system can be easily solved numerically to actually produce the resolvent  $K$ .

It is then a simple matter to show (using (2), (3), and 6)) that the function  $D(t, y; \lambda)$  satisfies the differential equation

$$D_\lambda(t, y; \lambda) = \frac{\int_0^1 D(t, y'; \lambda) D(y', y; \lambda) dy' - D(t, y; \lambda) \int_0^1 D(y', y'; \lambda) dy'}{D(\lambda)}, \tag{8}$$

and from (2), (4), and (7) we find that the initial condition for (8) is

$$D(t, y; 0) = k(t, y). \tag{9}$$

From the theory of complex variables [2], the number of zeros,  $N$ , of  $D(\lambda)$  (each

counted with proper multiplicity) in a closed contour  $C$ , with none on  $C$ , is given by

$$N = \frac{1}{2\pi i} \oint_C \frac{D_\lambda(\lambda)}{D(\lambda)} d\lambda. \quad (10)$$

We shall propose a numerical scheme for evaluating (10).

Assume that there are no roots on the unit circle. To find the number of eigenvalues enclosed by the unit circle in the  $\lambda$ -plane, we numerically integrate the initial value problem for the functions  $D(t, y; \lambda)$  and  $D(\lambda)$  in (3), (4), (8), and (9) from the origin, along a curve having no root of  $D(\lambda)$ , to  $\lambda = P$ , a point on the unit circle. In particular, this gives the values of  $D(P)$  and  $D_\lambda(P)$ . Then we adjoin the differential equation

$$\frac{d\Psi}{d\lambda} = \frac{1D_\lambda(\lambda)}{2\pi i D(\lambda)} = \frac{1}{2\pi i} \frac{-\int_0^1 D(y', y'; \lambda) dy'}{D(\lambda)}, \quad (11)$$

and the initial condition

$$\Psi(P) = \frac{1}{2\pi i} \frac{D_\lambda(P)}{D(P)}, \quad (12)$$

to the Cauchy system for  $D(t, y; \lambda)$  and  $D(\lambda)$  described previously. We now integrate this new system around the unit circle. It follows from (10) that the increment of  $\Psi$ , which must be zero or a positive integer, is the number of roots enclosed by the unit circle.

In an obvious manner, we can further localize the position of a root to various sectors, as when finding the roots of a polynomial numerically [6]. The formula

$$\frac{1}{2\pi i} \oint_C \lambda^k \frac{D_\lambda(\lambda)}{D(\lambda)} d\lambda = \sum_{i=1}^M \lambda_i^k, \quad k = 1, 2, \dots, \quad (13)$$

where  $\lambda_1, \lambda_2, \dots, \lambda_M$  are the roots enclosed by the unit circle, can be used to aid in the determination of both locations and multiplicities. Many obvious possibilities exist for finding the roots outside the unit circle.

### 3. NUMERICAL RESULTS

We shall present two numerical examples to illustrate the efficacy of the method. A Gaussian quadrature scheme of order  $N$  is used to approximate the integrals in (3) and (8) and a standard fourth-order Runge-Kutta scheme employing complex arithmetic is used to numerically integrate the resulting system of differential equations. For a quadrature scheme of order  $N$ , we get  $N^2$  differential equations

from (8) and one equation from each of (3) and (11). Additional equations may be added from (13) by letting

$$\frac{d\Omega}{d\lambda} \Omega_k = \frac{1}{2\pi i} \lambda^k \frac{D_\lambda(\lambda)}{D(\lambda)}, \quad k = 1, 2, \dots \tag{14}$$

As our first example we shall consider the integral equation

$$U(t) = \lambda \int_0^1 ixtU(x) dx, \quad 0 \leq t \leq 1. \tag{15}$$

Since this kernel is degenerate and symmetric it is easy to show that it has one complex eigenvalue at  $\lambda = -3i$ . For the numerical experiments we let  $N = 5$  and chose to integrate (3) and (8) from the origin along the positive imaginary axis, using a step size of  $\Delta r = 0.05$ , to the circle of unit radius. At this point we adjoined two additional differential equations for  $\Psi$  and  $\Omega_1$ . We then integrated, using a step size of  $\Delta\theta = \frac{1}{2}^\circ$ , counterclockwise around the unit circle. Upon returning to the point (0, 1) we obtained the following results for the increments in  $\Psi$  and  $\Omega_1$ ,

$$\begin{aligned} \Delta\Psi &= (1.8672 - 7, 8.1492 - 10), \\ \Delta\Omega_1 &= (-8.2814 - 10, 1.8892 - 7). \end{aligned}$$

indicating that the unit circle contains no zeros. The process was then repeated for a circle of radius four. The results were

$$\begin{aligned} \Delta\Psi &= (1.0000, 1.8623 - 9), \\ \Delta\Omega_1 &= (-3.0542 - 2, -2.9998). \end{aligned}$$

These results indicate that the circle of radius four encloses one eigenvalue and its position is approximately at  $\lambda = (0, -3)$ . These results could easily be improved upon by increasing the number of quadrature points and decreasing the integration step size. The second experiment was for the kernel

$$k(t, y) = e^{i\pi/4} e^{-i\pi(t-y)^2}. \tag{16}$$

This kernel is quite interesting in that it is symmetric but not Hermitian. Morgan [7] has shown that, for such kernels, certain classical techniques, such as Rayleigh-Ritz, may yield inaccurate results. Using the techniques discussed above (with  $N = 7$  and  $\Delta\theta = \frac{1}{2}^\circ$ ), we have calculated the first two eigenvalues to be

$$\begin{aligned} \lambda_1 &= (1.0648, -2.5679 - 3), \\ \lambda_2 &= (8.9473 - 1, -1.1703). \end{aligned}$$

Morgan does not list the eigenvalues, but the quantities  $w_i = 1 - |\mu_i|^2$ , where  $\mu_i = 1/\lambda_i$ . He then compares the results of an iterative scheme with the variational technique. We compare our results with those given by Morgan.

	$w_1$	$w_2$
variational	0.0607	0.2401
iterative	0.0800	0.2698
initial-value	0.0746	0.2650

#### 4. CONCLUSIONS

We have described a technique for calculating the complex eigenvalues for integral operations with general kernels. The method involves the combination of classical results of integral equations and complex variable theory with a recent technique for converting Fredholm integral equations into initial-valued differential equation systems. The technique is a flexible and accurate method for obtaining the complex eigenvalues of general kernels. The calculation of the eigenfunctions will be the subject of a future paper.

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